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# Development of Numerical Models for 1D and 2D Pollutant Transport Problems Using the Leapfrog Method with Additional Numerical Filters

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**ABSTRACT**

Pollutant transport in aquatic environments has become a critical environmental issue due to increasing contamination from industrial, domestic, and agricultural activities, which threaten ecosystems and human health. Understanding pollutant behavior through numerical modeling is essential for effective monitoring and mitigation strategies. This study aims to develop and evaluate a numerical model for simulating one- and two-dimensional pollutant transport using the advection–diffusion equation as the governing framework. The method employed is a finite difference approach based on the Leapfrog scheme combined with the Hansen numerical filter to improve model stability and simplicity. The model is tested through several benchmark cases, including pure advection, advection–diffusion, and Gaussian pulse scenarios in both one- and two-dimensional domains, with results compared to analytical solutions. The findings indicate that the proposed Leapfrog–Hansen model demonstrates good agreement with analytical solutions and achieves relatively small error values, particularly in one-dimensional cases. The results also show that spatial and temporal discretization significantly influence model stability and accuracy, where smaller step sizes generally improve performance but may reduce accuracy due to excessive filtering. In conclusion, the Leapfrog–Hansen model provides a simple yet effective alternative for pollutant transport simulation, especially for one-dimensional problems, with potential applications in environmental analysis and decision-making.

***Keywords:***
Advection ;
Diffusion ;
Hansen Filter ;
Leapfrog ;
Pollutant Transport.



## INTRODUCTION

Pollutant transport is the process of moving hazardous substances through various media, including water. The movement of pollutants in water involves two primary processes, namely advection and diffusion (Gomolka et al., 2022; Montazeri et al., 2022). Advection is described as a mechanism of substance transport induced by the movement of water masses (G. A. Putri et al., 2018). While diffusion is the mechanism of substance dispersion induced by concentration gradients (Simanungkalit & Magdalena, 2025). Each of these processes plays a crucial role in the spread of pollutants in the environment. Pollutants can originate from various human activities, such as industrial, domestic waste, and agricultural (Basuki et al., 2024; Fu et al., 2020; Johari et al., 2018). If not managed effectively, pollutants can threaten the ecosystems (Johari et al., 2018) and human health (Fu et al., 2020). Therefore, studying and understanding pollutant transport is crucial to reducing or minimizing these impacts. The study of pollutant transport plays a crucial role in environmental management. This study can serve as a tool for identifying pollutant transport and accumulation processes. The results of this identification will then help estimate the impact of pollutant distribution on the environment and ecosystems. Furthermore, a good understanding of pollutant distribution can be utilized as a tool for developing effective monitoring strategies, implementing mitigation measures, and informing policymaking related to pollutant management. An effective way to study pollutant distribution is by simulating pollutant transport using numerical models.

The advection-diffusion equation has long been used to simulate pollutant transport in waters (Halimi & Mat Isa, 2024; Maitsa et al., 2021; Parsaie & Haghiabi, 2017). Many researchers have previously developed numerical models capable of solving these equations, such as (Bahar et al., 2018; Gurarslan, 2021; Laek Sazzad Andallah, 2022; Sabran & Syafi'i, 2023). However, these models still have several shortcomings, such as low accuracy (Korkmaz & Dağ, 2012) and complex numerical algorithms (Dağ et al., 2011; I. El–Baghdady & S. El–Azab, 2016), resulting in long simulation times. In this study, an alternative numerical model will be developed to solve the advection-diffusion Equation as the governing Equation for pollutant transport. The model is designed based on the finite difference method in combination with the Hansen numerical filter. This model is hereinafter written as the Leapfrog-Hansen model. The Leapfrog method and the Hansen numerical filter are used due to their simplicity. Next, the Leapfrog-Hansen model is used to simulate one- and two-dimensional cases of pollutant transport. The simulation results are subsequently compared with available analytical solutions to evaluate the model's performance. Ultimately, the Leapfrog-Hansen model can serve as an alternative for modeling pollutant transport, which is beneficial in studying and mitigating the negative impacts of pollutant dispersion.

Recent studies retrieved through scholarly sources further show that pollutant transport modeling remains an active field of research. Halimi and Mat Isa (2024), for example, examined analytical solutions of the one-dimensional advection–diffusion equation with spatially dependent coefficients for instantaneous pollutant injection and found that variable velocity and diffusion significantly shape concentration distribution. Xuan et al. (2024) developed a numerical model for pollutant transport in vegetated rivers and demonstrated that more physically detailed modeling can effectively reproduce flow velocity and nutrient transport. More recently, Bey-Zekkoub et al. (2025) proposed an analytical–numerical framework for one-dimensional solute transport incorporating adsorption and degradation, while other 2025 studies explored simplified, comparative, and network-based transport formulations for rivers and coastal environments. Collectively, these studies confirm that the field continues to seek methods that are accurate enough for realistic applications but still computationally efficient and implementable.

Even so, an important research gap remains. Many recent models increase realism by incorporating vegetation, seepage, reaction terms, adsorption, or network complexity, but such advances often come with more elaborate numerical structures and higher implementation costs. At the same time, the manuscript's current literature framing indicates that earlier models may suffer from limitations related to accuracy, instability, or algorithmic complexity, especially when solving one- and two-dimensional advection–diffusion problems efficiently. This creates a methodological space for research that does not merely pursue higher physical complexity, but instead asks whether a relatively simple explicit finite-difference framework, strengthened by an additional numerical filter, can still produce stable and sufficiently accurate results across benchmark pollutant transport cases.

The urgency of this study therefore lies in both environmental need and computational practicality. Environmentally, worsening freshwater degradation and weak monitoring coverage increase the demand for predictive tools that can estimate pollutant behavior where observation networks are sparse. Computationally, environmental managers and researchers often need models that are transparent, reproducible, and not excessively burdensome to implement or calibrate. A numerical model that can achieve reasonable stability and accuracy with a simpler algorithm would be valuable for rapid scenario testing, classroom and research applications, and preliminary environmental assessment before more advanced modeling is undertaken.

The novelty of this research is located in its explicit integration of the Leapfrog finite-difference scheme with the Hansen numerical filter to construct a single modeling framework for both one-dimensional and two-dimensional pollutant transport problems. While many previous works have emphasized alternative finite-element, compact finite-difference, lattice Boltzmann, or random displacement approaches, this study offers a different contribution by revisiting a relatively simple time-marching scheme and strengthening it through numerical filtering. The novelty is therefore not only in the use of Leapfrog itself, but in testing whether the Leapfrog–Hansen combination can serve as a practical alternative that preserves computational simplicity while improving stability for pollutant transport simulation.

Based on that novelty, the purpose of this research is to develop and evaluate a numerical model capable of simulating pollutant transport phenomena in one- and two-dimensional domains using the advection–diffusion equation as the governing framework. More specifically, the study is intended to test the model under several benchmark cases, compare numerical outputs with analytical solutions, and assess how temporal and spatial discretization affect solution stability and accuracy. By doing so, the research contributes to the methodological discussion on how relatively simple numerical schemes can still be optimized for environmental transport modeling.

The contribution of this research is expected to be both theoretical and practical. Theoretically, it adds evidence to the numerical modeling literature by showing how filtering strategies can be combined with explicit finite-difference methods in advection–diffusion problems. Practically, it can support environmental analysis by providing an alternative simulation tool for pollutant transport in rivers and other water bodies, especially in contexts where monitoring data are limited and computational resources are constrained. Such a contribution is relevant because contemporary water-quality management increasingly depends on the integration of field data, analytical understanding, and predictive modeling.

Accordingly, the objectives of this study are to develop a Leapfrog–Hansen numerical model for solving one- and two-dimensional pollutant transport problems, to evaluate its stability and accuracy through comparison with analytical solutions, and to identify the influence of time-step and spatial-step selection on model performance. The benefits of the research are expected to include a clearer understanding of the strengths and limitations of the proposed scheme, an alternative reference for future numerical model development, and a practical basis for supporting pollutant monitoring, mitigation planning, and environmental decision-making. In this sense, the study is positioned not only as an exercise in numerical analysis, but also as a contribution to broader efforts to improve water-quality protection through better predictive tools.

## METHOD

### Governing Equation

In the two-dimensional model, transport is analyzed along two spatial directions, namely the $x$- and $y$-axes. Accordingly, pollutant movement is evaluated in both directions. Let $U$ and $V$ represent the flow velocities in both directions and let $D_x$ and $D_y$ represent the corresponding diffusion coefficients. Then, the pollutant concentration is denoted by $C$. Based of these definitions, the two-dimensional advection-diffusion Equation can be expressed as a differential equation as follows (Hutomo et al., 2019; Simanungkalit & Magdalena, 2025).

$$\frac{\partial C}{\partial t}+\frac{\partial(UC)}{\partial x}+\frac{\partial(VC)}{\partial y}=\frac{\partial}{\partial x}\left(D_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_y\frac{\partial C}{\partial y}\right) \tag{1}$$

In the two-dimensional advection-diffusion equation, $\frac{\partial C}{\partial t}$ denote the change in pollutant concentration over time, $\frac{\partial(UC)}{\partial x}$ and $\frac{\partial(VC)}{\partial y}$ denote the advective transport along two spatial axes. $\frac{\partial}{\partial x}\left(D_x\frac{\partial C}{\partial x}\right)$ and $\frac{\partial}{\partial y}\left(D_y\frac{\partial C}{\partial y}\right)$ describe the spread of pollutants due to the diffusion process along $x$-axis and $y$-axis.

**Leapfrog-Hansen Model**

The Leapfrog method is an explicit finite difference method that can be used to solve differential equations, especially in simulations involving time-dependent problems, such as pollutant transport. The name "Leapfrog" comes from the way this method solves derivatives with respect to time. The Leapfrog method uses a second order accurate central difference scheme to solve time derivatives (Aluthge et al., 2016; Sukhinov et al., 2021). A similar scheme is also used to solve spatial derivatives. The following are the results of solving the governing equation using the Leapfrog method.

$$\begin{aligned} C_{i,j}^{n+1} = C_{i,j}^{n-1} - 2\Delta t\Bigg(&\frac{U_{i+1,j}^{n}C_{i+1,j}^{n}-U_{i-1,j}^{n}C_{i-1,j}^{n}}{2\Delta x}+\frac{V_{i,j+1}^{n}C_{i,j+1}^{n}-V_{i,j-1}^{n}C_{i,j-1}^{n}}{2\Delta y} \\ &- D_x\frac{C_{i+1,j}^{n}-2C_{i,j}^{n}+C_{i-1,j}^{n}}{\Delta x^2}-D_y\frac{C_{i,j+1}^{n}-2C_{i,j}^{n}+C_{i,j-1}^{n}}{\Delta y^2}\Bigg) \end{aligned} \tag{2}$$

This study also uses a numerical filter, a stabilization scheme that enhances the numerical model's stability. Specifically, the Hansen numerical filter is employed. The main advantage of the Hansen filter lies in its simple algorithm, making it easy to implement. This filter updates the parameter values in the governing Equation at each computation point in each iteration using the following Equation.

$$C_{filter} = C_f \times C_{i,j} + \left(1-C_f\right)\left(\frac{C_{i+1,j}+C_{i-1,j}+C_{i,j+1}+C_{i,j-1}}{4}\right) \tag{3}$$

In this formulation, $C_f$ denotes a correction factor which ranges between 0 and 1, $C$ denotes the pollutant concentration, and $C_{filter}$ is the updated pollutant concentration. In this study a value of 0.99 is used for $C_f$, as it has been shown in previous studies to provide stable results (Hafiyyan, Adityawan, et al., 2021; Hafiyyan et al., 2023; Hafiyyan, Harlan, et al., 2021; Maitsa et al., 2020; Mellivera et al., 2020; Pratama et al., 2025; P. I. D. Putri et al., 2020).

## RESULTS AND DISCUSSION

### 1D Pure Advection

The first case used in this study is a pollutant transport case involving only advection. Several previous researchers have also studied this case. In this case, the channel is assumed to be 9,000 m long, and the flow velocity is 0.5 m/s. Furthermore, the channel is considered to have a constant and flat cross-sectional shape. For this case, the equations used for initial conditions, boundary conditions, and analytical solutions can be seen in.

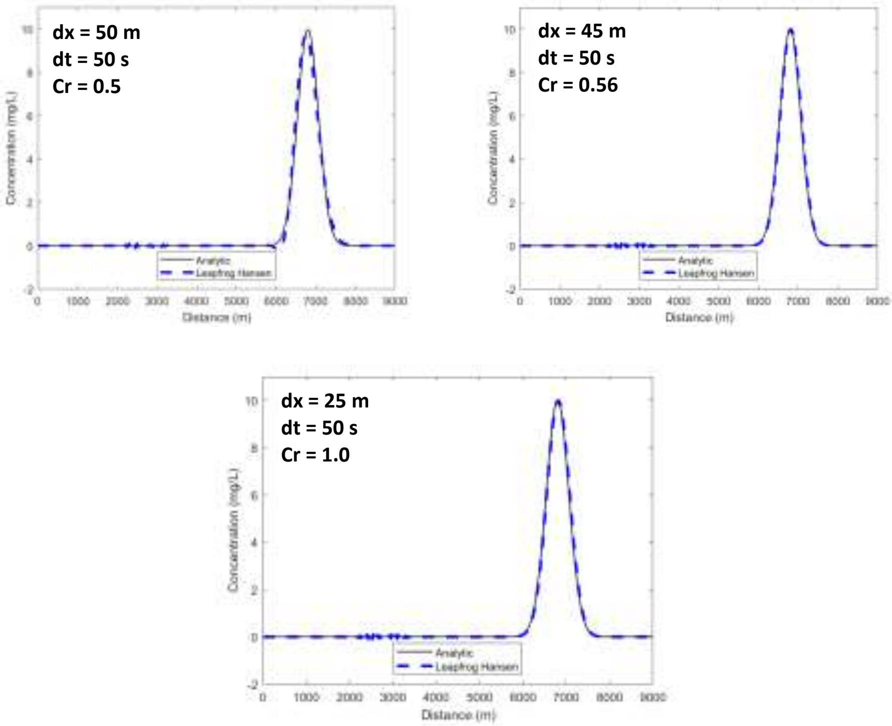


**Figure 1.** Simulation results of 1D pure advection cases with spatial step variations at t = 9600s

The results obtained at a simulation time of 9600 s using different spatial step sizes are shown in Figure 1. The figure indicates that the Leapfrog-Hansen model becomes increasingly stable as the spatial step decreases. However, it should be noted that the Leapfrog model becomes unstable when the Courant number exceeds one. The spatial step also affects the accuracy of the Leapfrog-Hansen model in modeling pollutant transport. The effect is evident in the error values for various spatial step sizes, as presented in Table 1. Based on the table, the smaller the spatial step, the smaller the error value and the better the accuracy.

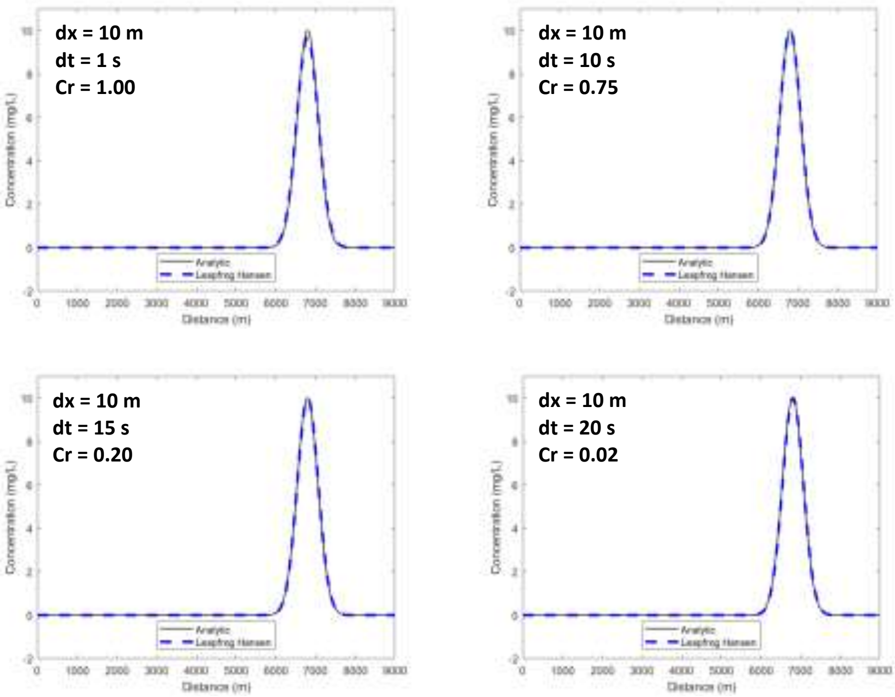


**Figure 2.** Simulation results of 1D pure advection cases with time step variations at t = 9600s

Next, simulations were conducted with various dt values to determine the effect of time step on the simulation results (Figure 2). The results indicate that the Leapfrog-Hansen model becomes more stable with smaller time steps. However, too small a time step will result in decreased model accuracy, as shown in Table 1. The decrease is due to the Hansen numerical filter, which averages the concentration values around the filtered point. A smaller time step results in more frequent filtering and more uniform concentration values, ultimately leading to a decrease in accuracy. Therefore, determining the appropriate time step is crucial in the Leapfrog-Hansen model.

**Table 1.** Absolute error of the leapfrog-hansen model for the 1D pure advection case

| Time Step, dt (s) | Spatial Step, dx (m) | Courant Number, Cr | Mean Absolute Error | Time Step, dt (s) | Spatial Step, dx (m) | Courant Number, Cr | Mean Absolute Error |
|---|---|---|---|---|---|---|---|
| 20 | 10 | 1.00 | 0.012 | 50 | 50 | 0.50 | 0.087 |
| 15 | | 0.75 | 0.008 | | 45 | 0.56 | 0.065 |
| 10 | | 0.50 | 0.005 | | 25 | 1.00 | 0.036 |
| 1 | | 0.05 | 0.024 | | 20 | 1.25 | Unstable |

In this study, the Leapfrog–Hansen simulation results are further evaluated in comparison with results from other numerical models reported in prrevious studies. The values used for this comparison are the norm error values of each model, as shown in Table 2. The results show that the Leapfrog-Hansen model yields a smaller norm error value compared to other numerical models, including those using the Finite Element Method with Linear Shape Functions (FEMLSF) and Finite Element Method with Quadratic Shape Functions (FEMQSF). Based on this comparison, it is evident that the Leapfrog-Hansen model yields more accurate results compared to previous studies that employed models with more complex algorithms, such as the finite element method.

**Table 2.** Comparison of norm error between the proposed model and other numerical models (Cr = 0.5)

| **FEMLSF** (Dağ et al., 2006) | | **FEMQSF** (Dağ et al., 2006) | | **Leapfrog-Hansen** | |
|---|---|---|---|---|---|
| $\mathbf{L_2}$ | $\mathbf{L_\infty}$ | $\mathbf{L_2}$ | $\mathbf{L_\infty}$ | $\mathbf{L_2}$ | $\mathbf{L_\infty}$ |
| 7.894 | 0.380 | 7.908 | 0.373 | 0.352 | 0.051 |

**1D Advection-Difussion**

The second case is a one-dimensional advection-diffusion case. This case is used to assess the performance of the Leapfrog-Hansen model in simulating a more complex case than the first case. Pollutant transport in this case involves not only advection but also diffusion. The channel length for this case is 100 m. Furthermore, the flow velocity ($U$) and dispersion coefficient ($D$) are 0.01 m/s and 0.0002 $m^2$/s, respectively. For this case, the equations used for initial conditions, boundary conditions, and analytical solutions can be seen in (Gurarslan et al., 2013).

The simulation results at t = 3600 s are presented in Figure 3 and Figure 4. Figure 3 compares the exact solution and the Leapfrog–Hansen results for different time steps at dx = 0,25 m. As shown in the figure, the Leapfrog–Hansen model exhibits good agreement with the exact solution. Additionally, Figure 3 shows that the smaller dt value causes a decrease in the accuracy of the proposed model. The reduction in accuracy can also be observed from the error rates of each variation (Table 3), where the mean absolute error values are 0.011 for dt = 0.08s, 0.014 for dt = 0.06s, and 0.020 for dt = 0.04s. In addition, Figure 4 presents a comparison of the exact solution and the Leapfrog–Hansen simulation results under different spatial step sizes (dx). The results show that the smaller the value of dx, the better the agreement with the exact solution. The spatial step also affects the accuracy of the Leapfrog-Hansen model, as shown in Table 3. The table shows that the smaller the dx value, the smaller the error rate or the better the accuracy.

**Table 3.** Absolute error of the leapfrog-hansen model for the 1D advection-diffusion case

| **Time Step, dt (s)** | **Spatial Step, dx (m)** | **Mean Absolute Error** | **Time Step, dt (s)** | **Spatial Step, dx (m)** | **Mean Absolute Error** |
|---|---|---|---|---|---|
| 0.08 | 0.25 | 0.011 | 0.01 | 1.00 | 0.239 |
| 0.06 | | 0.014 | | 0.40 | 0.092 |
| 0.04 | | 0.020 | | 0.10 | 0.014 |

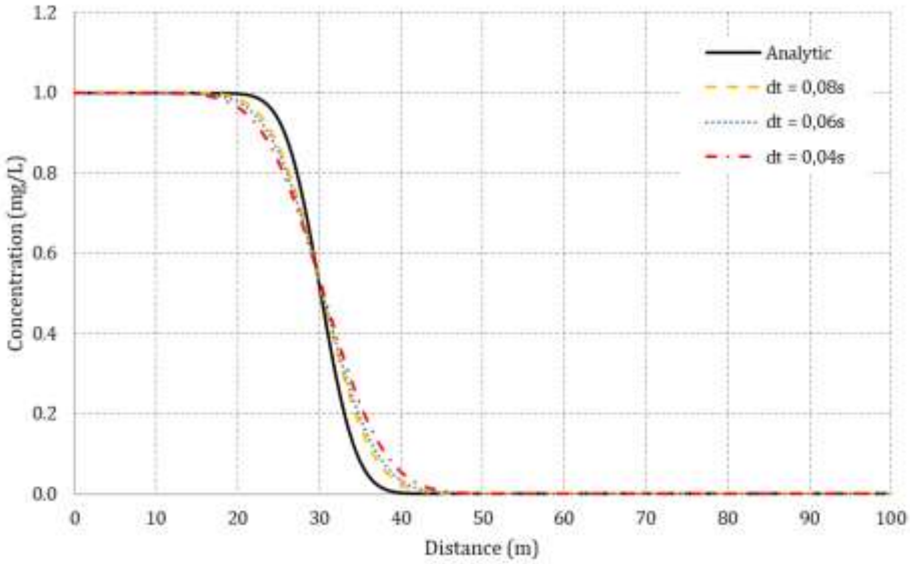


**Figure 3.** Simulation results of second cases with dx = 0.25m at t = 3000s

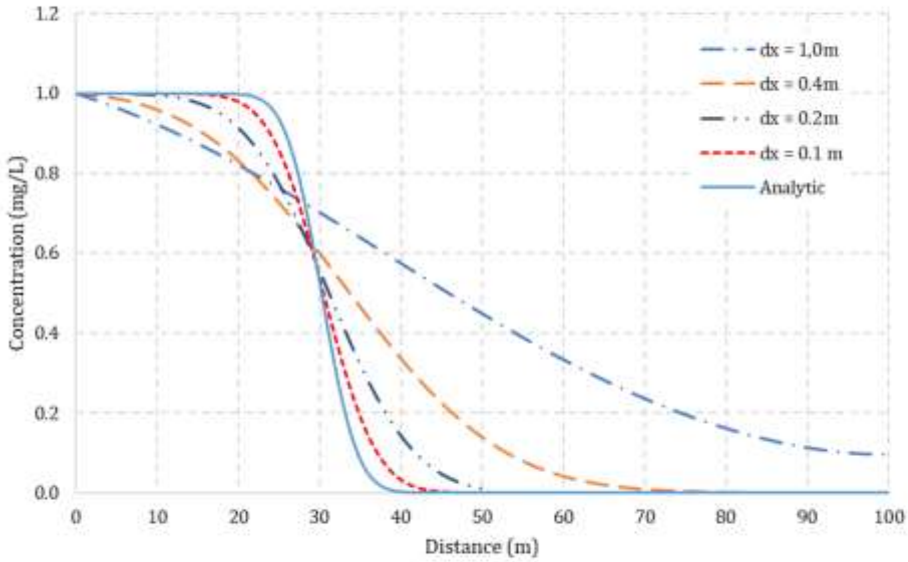


**Figure 4.** Simulation results of second cases with dt = 0.01s at t = 3000s

**1D Gaussian Pulse**

The one-dimensional Gaussian Pulse case is the third case to be simulated in this study. A Gaussian pulse is a common initial condition for the advection-diffusion problems. This case is commonly used to model how the concentration profile spreads over time due to mass motion (advection) and random motion (diffusion). The Gaussian Pulse case, with its bell-shaped Gaussian curve, is widely used because it can be solved analytically. The presence of this analytical solution can be used to test the accuracy of numerical methods in solving the advection-diffusion equation. In this case, the flow velocity (U) is 0.8 m/s and the dispersion coefficient (D) is 0.005 $m^2$/s. Then, the equations used for initial conditions, boundary conditions, and analytical solutions can be seen in (Gurarslan et al., 2013).

The simulation results at a simulation time of 5 s with variations in dt and dx are shown in Figure 5 and Figure 6 respectively. Figure 5 illustrates that the Leapfrog-Hansen model yields a pollutant concentration pattern comparable to the exact solution. The difference between the Leapfrog-Hansen model results and the exact solution lies in the height of the pollutant peak. The largest time step variation used in this case is 0.00005 s, because simulations with time steps greater than 0.00005 s cannot produce results due to numerical instability. Figure 5 illustrates that the smaller the time step, the greater the deviation of the results from the exact solution and the less accurate they become. This decrease in accuracy is also evident in Table 4. This case also demonstrates the same principle as the previous two cases, namely that determining the right time step is crucial for providing stable results and good accuracy.

Furthermore, a comparison of the simulation results of the Leapfrog-Hansen model with the exact solution, using various variations of dx, is presented in Figure 6. The results show that the best agreement occurs in the simulation using the smallest dx. Additionally, the simulation with the smallest dx also yields the smallest error rate, as presented in Table 4. The table indicates that to obtain good and accurate simulation results, it is necessary to reduce the spatial step (dx).

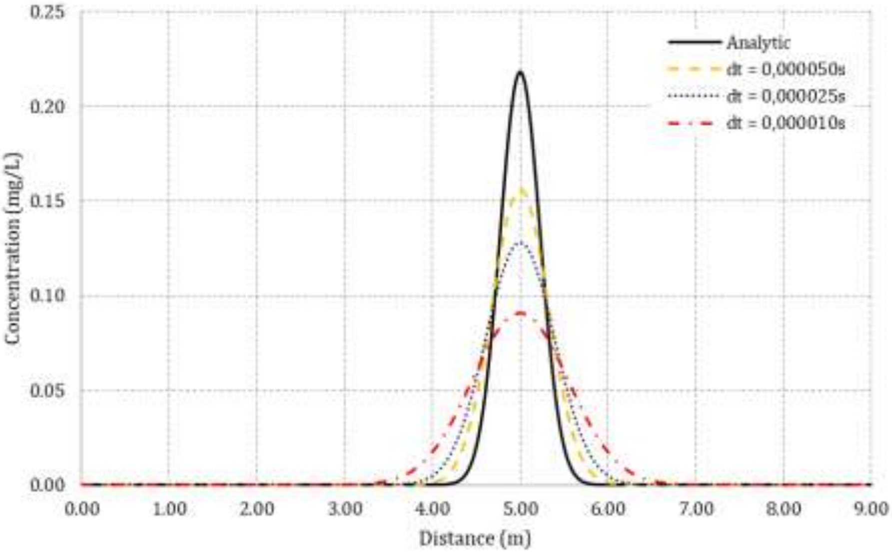


**Figure 5.** Simulation results of 1D Gaussian pulse cases with time step variations (dt) and dx = 0.01m at t = 5s

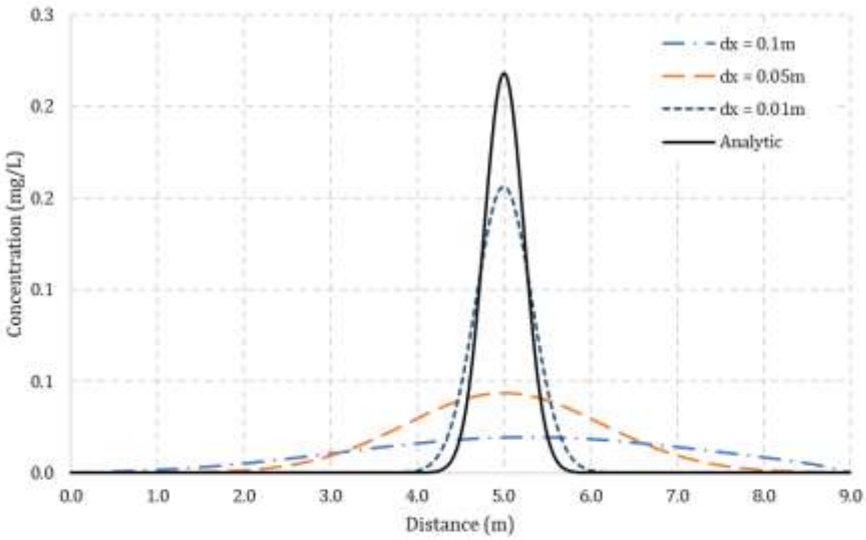


**Figure 6.** Simulation results of 1D Gaussian pulse cases with spatial step variations (dx) and dt = 0.00005s at t = 5s

**Table 4.** Absolute error of the leapfrog-hansen model for the 1D Gaussian pulse case

| Time Step, dt (s) | Spatial Step, dx (m) | Mean Absolute Error | Time Step, dt (s) | Spatial Step, dx (m) | Mean Absolute Error |
|---|---|---|---|---|---|
| 0.000050 | | 0.004 | | 0.10 | 0.019 |
| 0.000025 | 0.01 | 0.007 | 0.000010 | 0.05 | 0.018 |
| 0.000010 | | 0.011 | | 0.01 | 0.004 |

### 2D Gaussian Pulse

The last case is the two-dimensional Gaussian Pulse case. This case is used to see how the Leapfrog-Hansen model performs in simulating a two-dimensional pollutant transport case. The computational domain for this case has a length of 2 m and a width of 2 m. Furthermore, the flow velocities in both the $x$-direction ($U$) and $y$-direction ($V$) are 0.8 m/s, while the dispersion coefficients in the $x$ and $y$ directions are identical, each equal to 0.01 $m^2/s$. In the initial condition, the Gaussian Pulse is located at x = 0.5 m and y = 0.5 with a height of 1 $m^2/s$. For this case, the equations used for initial conditions, boundary conditions, and analytical solutions can be seen in (Gurarslan et al., 2013).

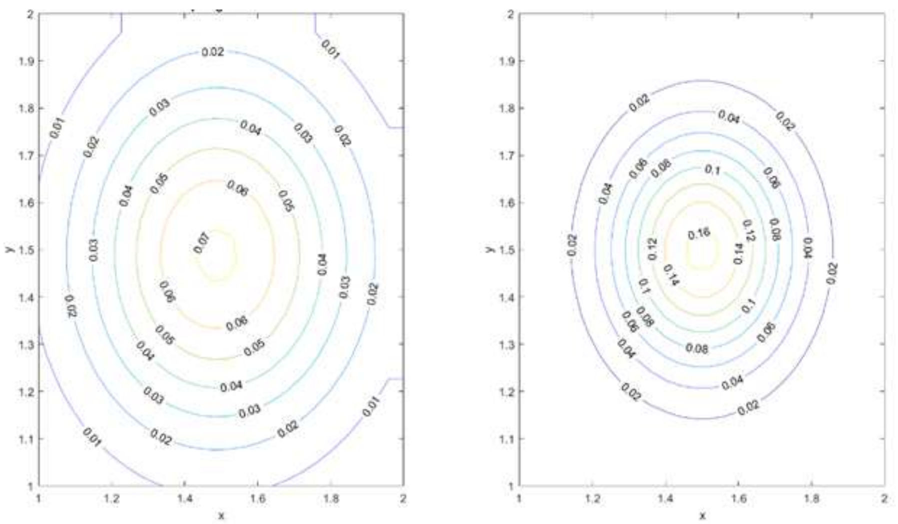

**Figure 7.** Pollutant concentration contours for Leapfrog-Hansen (left) and analytical (right)

In this case, the simulation was conducted using the initial and boundary conditions described above. The simulation was performed by trial and error on the time and spatial step values to obtain optimal results. After trial and error, optimal results were obtained when the time step was 0.000125 s and the spatial step in the x- and y-directions was 0.04 m. A comparison of the Leapfrog-Hansen model results with the exact solution at t = 1.25 s is shown in Figure 7. Based on the figure, the pollutant distribution obtained from the Leapfrog–Hansen model shows good agreement with the exact solution. Nevertheless, a difference is observed in the peak pollutant concentration. Specifically, the maximum concentration predicted by the Leapfrog–Hansen model is 0.072 mg/L lower than that of the analytical solution, which reaches 0.164 mg/L. This value also did not outperform the results from previous numerical models (Gurarslan et al., 2013; Maitsa et al., 2021). In addition, differences also occur in the speed of pollutant spread. In the Leapfrog-Hansen model, pollutants spread faster than in the analytical solution. The differences can be observed from the distribution of pollutants in the Leapfrog-Hansen model at t = 1.25 s, which has reached the domain wall. In contrast, the distribution of pollutants in the exact solution has not yet reached the wall of the domain.

## CONCLUSION

This study developed a numerical model to simulate the transport of pollutants in one- and two-dimensional. The model, known as the Leapfrog-Hansen model, was developed using the Leapfrog method and incorporating the Hansen filter. The Leapfrog-Hansen model was then used to simulate several cases of one- and two-dimensional pollutant transport. The results showed that the Leapfrog-Hansen model could simulate pollutant transport phenomena, both one- and two-dimensional. The results also indicated that the pollutant transport pattern matches well with the analytical solution. Additionally, the one-dimensional Leapfrog-Hansen model's accuracy is quite good, with a relatively small error value compared to analytical solutions and other numerical models from previous studies. However, the leapfrog-hansen model shows relatively large differences in pollutant peaks in the two-dimensional case. The study also demonstrated that the time step and spatial step significantly influence the Leapfrog-Hansen model. Generally, smaller time steps and spatial steps will yield better model stability and accuracy. Nevertheless, a too-small time step also decreases the model's accuracy. The decrease is due to the way the Hansen filter works, which uses the average parameter value from the surrounding computational points. As the time step decreases, the averaging process becomes more frequent, and the concentration values become more uniform. Then, the concentration values drift away from the analytical solution. Therefore, determining the time step in the Leapfrog-Hansen model is crucial for obtaining optimal simulation results. The findings of this study suggest that the Leapfrog–Hansen model can serve as an alternative numerical approach for modeling pollutant transport, especially one-dimensional case. For further studies, the model can be developed using higher-order Hansen filters to obtain more accurate simulation results, especially in the two-dimensional case.

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